Mpi and the Sieve of Eratosthenes

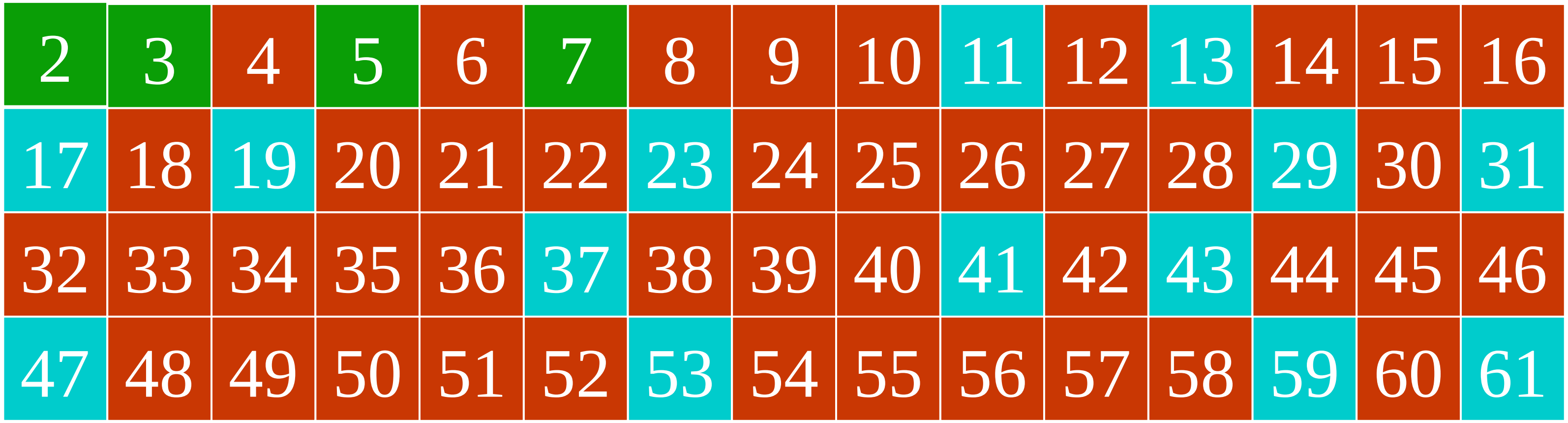
Outline:

- Sequential algorithm
- Sources of parallelism
- Data decomposition options
- Parallel algorithm development, analysis
- MPI program
- Benchmarking
- Optimizations

Sequential algorithm for finding primes

```
1. Create list of unmarked natural numbers 2, 3, …, $n$
2. $k \leftarrow 2$
3. Repeat
    (a) Mark all multiples of $k$ between $k^2$ and $n$
    (b) $k \leftarrow$ smallest unmarked number $> k$ until $k^2 > n$
4. The unmarked numbers are primes
```

Representation of algorithm



2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
47	48	49	50	51	52	53	54	55	56	57	58	59	60	61

Complexity: $\Theta(n \ln \ln n)$

Identify what can be parallelized

- Domain decomposition – what is the domain? Represent the data as an array of integers.
  - Divide data into pieces
  - Associate computational steps with data
- One primitive task per array element
- The tasks in 3(a) and 3(b) need to be analyzed

First, consider the tasks in 3(a)

Mark all multiples of k between k^2 and n

In pseudocode for the sequential algorithm, this can be written as:

```
for all $j$ where $k^2 \leq j \leq n$ do
    if $j$ mod $k = 0$ then
        mark $j$ (it is not a prime)
    endif
endfor
```

In the parallel case, j is an element of an array and represents a task

And then, consider the tasks in 3(b)

Find smallest unmarked number > *k*

This step ignores the marked array elements, so the number of tasks has been reduced. Can be accomplished by:

- Perform a reduction to find the smallest unmarked number > *k*
- Broadcast the result to all processes

Agglomeration of the tasks

- Consolidate tasks – each iteration of the sieve algorithm reduces the number of elements to consider.
- Reduce communication cost – current value of *k* needs to be shared with all processes.
- Balance computations among processes – as the calculation proceeds, less tasks remain with smaller indices.

How to divide up the data

- Interleaved (cyclic) – if n tasks and p processes, a process is given, tasks are assigned “round robin”
  - Easy to determine “owner” of each index
  - Leads to load imbalance *for this problem*
- Block decomposition – each process is given a contiguous block of tasks
  - Balances loads
  - More complicated to determine owner if *n* not a multiple of *p*

Load balance problem in interleaved division of data

Consider p = 4, so

p_0 has tasks with values 2, 6, 10, 14, 18, …

p_1 has tasks with values 3, 7, 11, 15, 19, …

p_2 has values 4, 8, 12, 16, 20, ...

p_3 has values 5, 9, 13, 17, 21, …

Processes p_0 and p_2 have no more tasks after the case $k = 2$.

How does block decomposition work?

- Want to balance workload when *n*, the number of tasks, is not a multiple of *p*, the number of processes
- Each process gets either ceil(*n/p)* or floor(*n/p)* elements
- Seek simple expressions to identify task and process
- Find low, high indices given a process number
- Find the process given an array index

First approach to block decomposition

- Let r = n mod p
- If r = 0, all blocks have same size and it is straighforward to find which array elements belong to which process
- Else
  - First r blocks have size ceil(n/p)
  - Remaining p-r blocks have size floor(n/p)

When r != 0

First element controlled by process i:

$j = i*\text{floor}(n/p) + \min(i,r)$

Last element controlled by process i:

$j = (i+1)*\text{floor}(n/p) + \min(i+1,r) - 1$

Process, q, controlling element j:

$q = \min(\text{floor}(j/(\text{floor}(n/p)+1),\text{floor}(j-r)/\text{floor}(n/p)))$

Some examples using the first approach

17 elements divided among 7 processes

17 elements divided among 5 processes

17 elements divided among 3 processes

Second approach – scatter larger blocks among smaller blocks

17 elements divided among 7 processes

17 elements divided among 5 processes

17 elements divided among 3 processes

Assigning indices to processes in second approach

First element controlled by process i:

$j = floor(i*n/p)$

Last element controlled by process i:

$j = floor((i+1)*n/p)-1$

Process controlling element j:

$q = ceil((p*(j+1)-1)/n)$

Macros to program the second approach

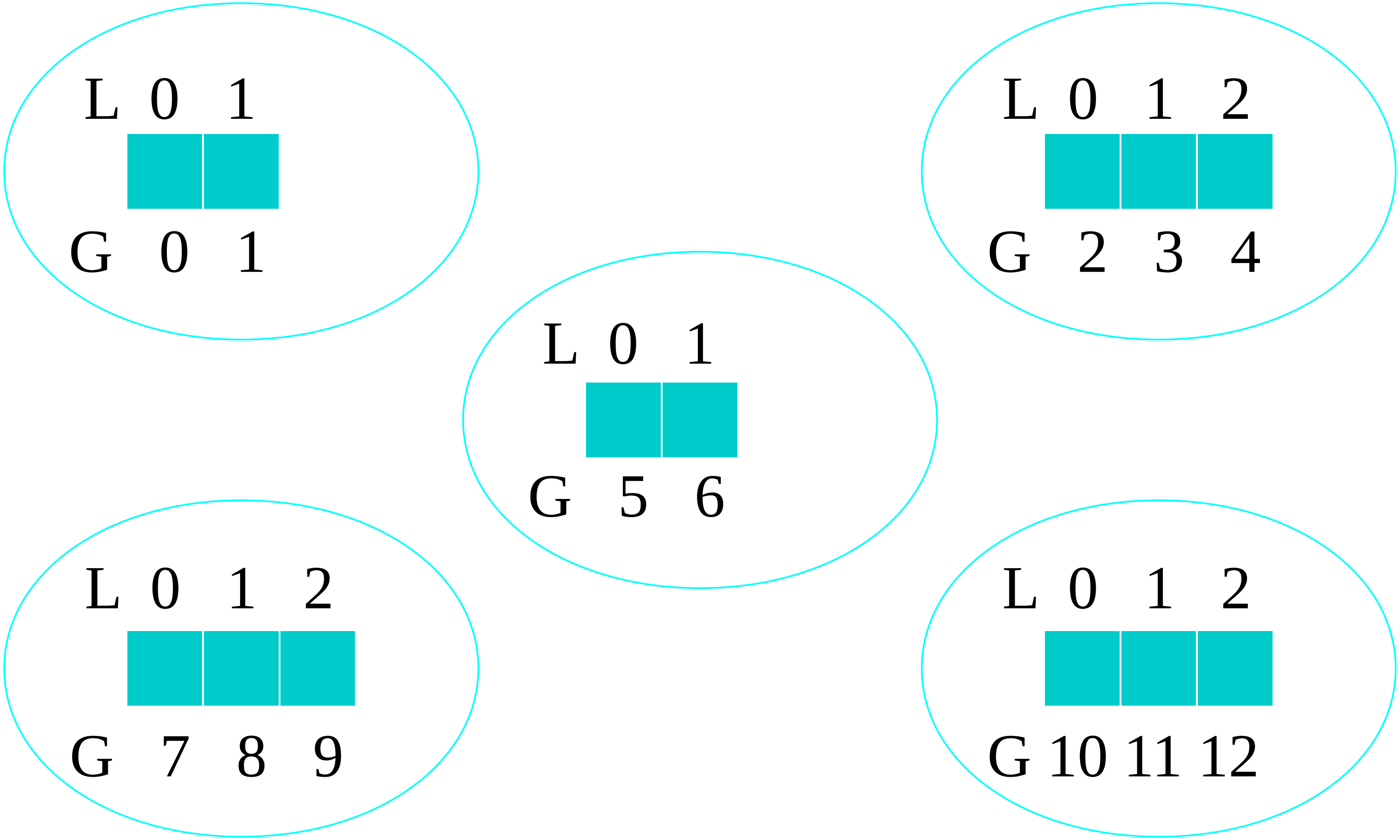
```
#define BLOCK_LOW(id,p,n)  ((i)*(n)/(p))


#define BLOCK_HIGH(id,p,n) (BLOCK_LOW((id)+1,p,n)-1)


#define BLOCK_SIZE(id,p,n) \
                     (BLOCK_LOW((id)+1)-BLOCK_LOW(id))


#define BLOCK_OWNER(index,p,n) (((p)*(index)+1)-1)/(n))
```

Each process has local variables that correspond to sequential variables



Comparing the indices in the sequential code with the parallel code

- Sequential program

```
for (i = 0; i < n; i++) {
   …
}
```

Local index *i* on this process…

- Parallel program

```
size = BLOCK_SIZE (id,p,n);
for (i = 0; i < size; i++) {
   gi = i + BLOCK_LOW(id,p,n);
   ...
}
```

…takes place of sequential program’s index *i*

The method of decomposition affects the implementation

- The largest prime used in the algorithm to remove multiples is $\sqrt{n}$
- The first process has floor(n/p) elements
- The algorithm finds all possible primes if $p < \sqrt{n}$
- The first process always broadcasts the next sieving prime
- No reduction step is needed

Fast marking of rejected elements

Block decomposition allows same marking as sequential algorithm:

mark elements j, $j + k$, $j + 2k$, $j + 3k$, …

instead of

```
for all $j$ in block
    if $j \bmod k = 0$ then mark $j$      //it is not a prime
```

Parallel Algorithm Development

1. Create list of unmarked natural numbers 2, 3, …, n

Each process creates its share of list

2. $k \leftarrow 2$

Each process does this

3. Repeat

Each process marks its share of list

   (a) Mark all multiples of k between k^2 and n

   (b) $k \leftarrow$ smallest unmarked number $> k$

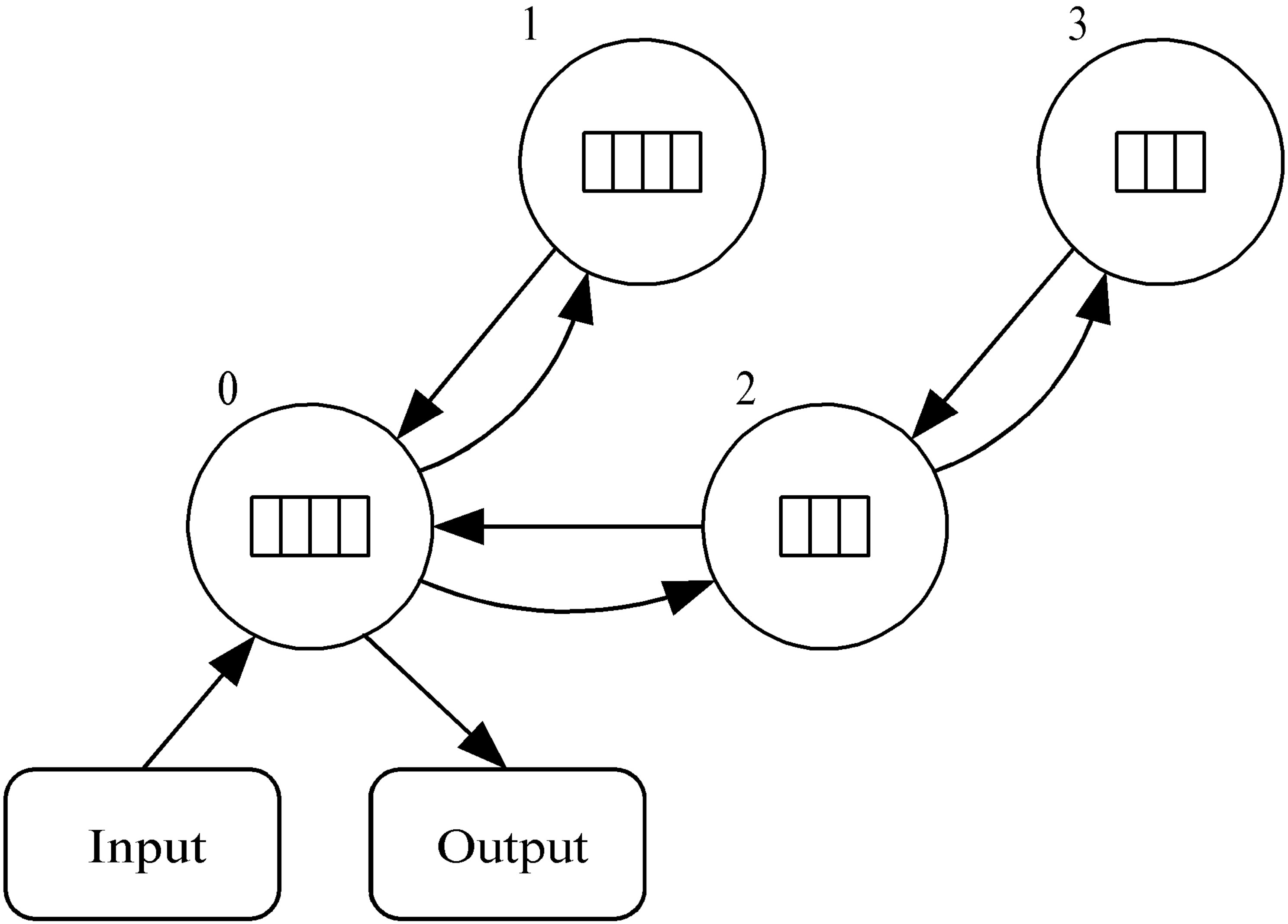
Process 0 only

   (c) Process 0 broadcasts k to rest of processes until $k^2 > n$

4. The unmarked numbers are primes

5. Reduction to determine number of primes

Task/Channel Graph



How to broadcast data from one process to another

```
MPI_Bcast (&k, 1, MPI_INT, 0, MPI_COMM_WORLD);


int MPI_Bcast (
   void *buffer, /* Addr of 1st element */
   int count,    /* # elements to broadcast */
   MPI_Datatype datatype, /* Type of elements */
   int root,     /* ID of root process */
   MPI_Comm comm)  /* Communicator */
```

Some Improvements to the Algorithm

1. Delete even integers

- Cuts number of computations in half
- Frees storage for larger values of *n*

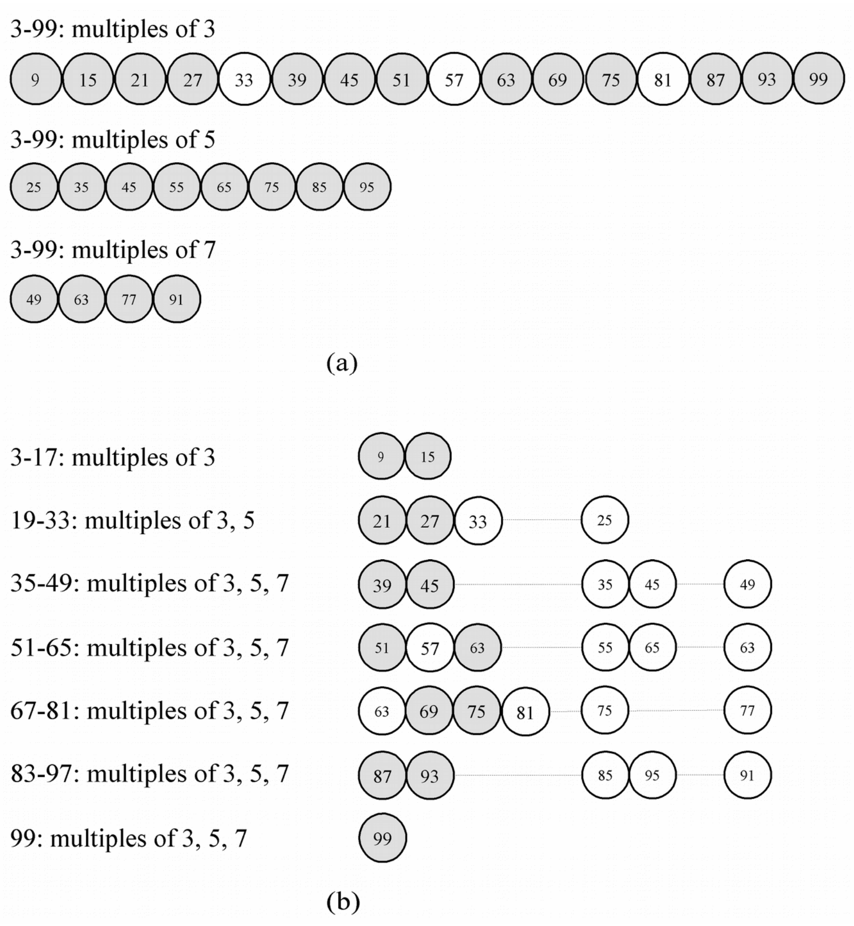
2. Each process finds own sieving primes

- Replicating computation of primes to $\sqrt{n}$
- Eliminates broadcast step

3. Reorganize loops

- Exchange the do-while and the for loop
- Increases cache usage

Reorganizing the code by inverting the loops



(a) Lower cache hit rate (usage) of the original arrangement of the loops

(b) Higher cache hit rate when the loops are exchanged.

Summary of content

- Sieve of Eratosthenes: parallel design uses domain decomposition
- Compared two block distributions
  - Chose one with simpler formulas
- Introduced **`MPI_Bcast()`** for communication
- Optimizations reveal importance of maximizing single-processor performance when using MPI