Some Basic Set Theory

Eric Pacuit
Brooklyn College, CUNY
www.cs.gc.cuny.edu/∼epacuit
epacuit@cs.gc.cuny.edu

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1 Basic Set Theory

1.1 Introduction

We often groups things together. Everyone in this class, your group of friends, your family. These are all collections of people. When we study set theory we are studying collections of things. Now, "things" is a very general word. What types of things are we looking at. The simple answer is that we don't care what type of things we are looking at. What we are interested in is whether a certain object (or thing) is *part of* a group. A set is one of these collections. An object in a group or set is said to be an element of the set. We will look at various operations on sets, I.e. how sets interact with one another.

1.2 Definitions

The easiest way to visualize a set is to consider a Venn Diagram. A Venn Diagram is basically a geometrical interpretation of a set. For now a set is just a collection of objects that have something in common. However there is one problem. Before we can talk about any set we must first consider were this set exists. In other words, where did its elements come from. Think about it — if we didn't have a human race then we could not begin to talk about the set of students in this class or the set of people that live in Brooklyn, if we did not have all the integers we could not begin to talk about the set of 2 digit numbers or the set of prime numbers. So in order to talk about a set, we must first define a universe.

Definition Idea 1 (Universal Set) [1] *The* **Universal Set** *is a collection of all the items or objects in question, I.e. it is a collection of all possible elements.*

Definition Idea 2 (Element) *A member of a set.*

An element is represented by a dot.

Definition Idea 3 (Set) *Any collection of elements from the universal set.*

A set is represented by a circle (inside the box which is the universal set).

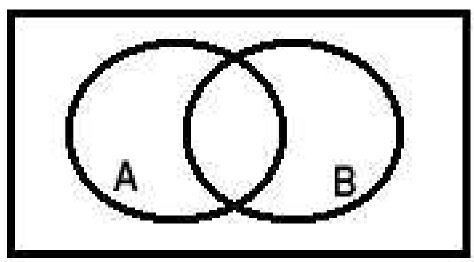

Figure 1: Two sets A and B

Definition Idea 4 (Subset) *A* **Subset** *is a sub collection of a set.*

Notice that every set is a subset of the Universal Set. The notion of subset can be represented in figure 2.

Definition Idea 5 (Union) *The* **Union** *of two (or more) sets is a set that contains all the elements of each set.*

For two sets the shaded area represents the union in figure 3.

[1] Technically speaking, there is no set that contains all sets, a so called universe of sets. The set of all sets fails to be a set because of Russell's paradox, but this is beyond the scope of these notes, and so I will not discuss it any further. Additional information can be found in any set theory book.

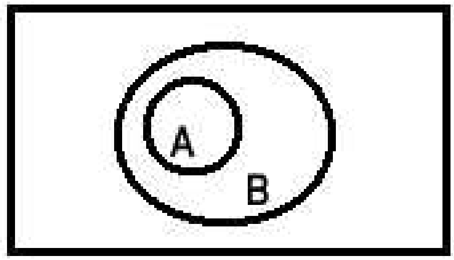

Figure 2: A is a subset of B

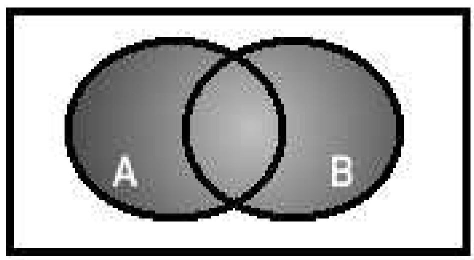

Figure 3: The union of A and B

Definition Idea 6 (Intersection) *The* **Intersection** *of two (or more) sets is the set of all items in common to each set.*

The shaded area represents the intersection in figure 4.

Figure 4: The intersection of A and B

Definition Idea 7 (Complement) *The* **Complement** *of a set is the set of all elements not contained in that set.*

Why is the notion of a universal set necessary for this definition? The shaded area in figure 5 represents the complement of a set .

Definition Idea 8 (Set Difference) *The difference between two sets A and B (A minus B) is all elements in A but not in B.*

The shaded area in figure 6 represents the difference of two sets.

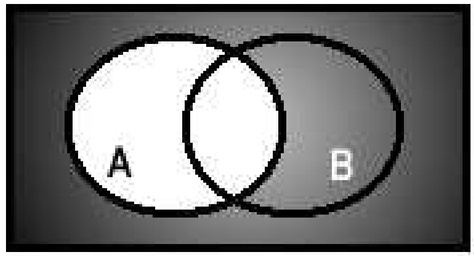

Figure 5: The complement of A

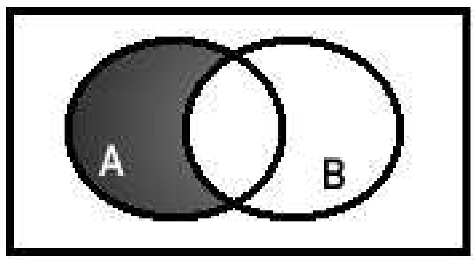

Figure 6: A minus B

Definition Idea 9 (Symmetric Difference) *The symmetric difference of two sets is all the elements in either set* but not in both.

Definition Idea 10 (Product) *The* **Cartesian Product** *of two or more sets is all possible pairings of each elements of the sets.*

Definition Idea 11 (Power Set) *The* **Power Set** *of a set is the set of all subsets of a set.*

Definition Idea 12 (Null Set) *The null or empty set is a set that contains no elements.*

Notice that the empty set is a subset of every set.

1.3 Notation

This part contains the notational definitions. Think of this section as the language which will allow you to write down the ideas and notions you understand from

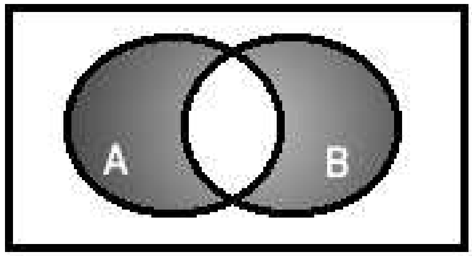

Figure 7: The symmetric difference of A and B

the previous section. We will represent sets by capital letters, and elements of sets will be represented by lower case letters. We have two ways to write down the contents of a set:

1. List all the elements of the set. Each element should be separated by a comma and contained between curly brackets ($\{\}$). For example suppose A is the set of the first 5 letters of the alphabet. Then $A = \{a, b, c, d, e\}$.

2. Write down a property that **all** elements of the set have in common. For example if A is the set of all positive integers, then $A = \{x | x \geq 0 \text{ and } x \text{ is an integer}\}$. This is read "$x$ such that x is greater than or equal to zero and x is an integer".

Suppose A and B are two sets.

Definition 1 (Universal Set) *The* **Universal Set** *will be represented by the letter U.*

Definition 2 (Element) *If we want to say x is an element of A, then we write $x \in A$*

Definition 3 (Union) $A \cup B = \{x | x \in A \textit{ or } x \in B\}$

Definition 4 (Intersection) $A \cap B = \{x | x \in A \textit{ and } x \in B\}$

Definition 5 (Complement) $\overline{A} = A^{\mathrm{C}} = \{x | x \notin A\}$

Definition 6 (Set Difference) $A \setminus B = A - B = \{x | x \in A \textit{ and } x \notin B\}$

Definition 7 (Symmetric Difference) $A \triangle B = (A - B) \cup (B - A)$

Definition 8 (Cartesian Product) $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

Definition 9 (Empty Set) *The set with no elements is denoted by* $\varnothing$

Definition 10 (Power Set) *The poser set of a set* A *is the set of all subsets of* A*,* including S and $\varnothing$*, and is denoted* 2^A *or* $\mathcal{P}(A)$*. So* $2^A = \{B \mid B \subseteq A\}$

Definition 11 (Cardinality of a Set) *The cardinality of a finite set* A *is the total number of elements in* A*, and is denoted* $|A|$*.*

Definition 12 (Partition) *A* **partition** *of a set* S *is a collection of sets* $\mathcal{S} = \{S_1, S_2, \ldots\}$ *(possibly infinite) such that*

- *the sets are* **pairwise disjoint***, that is* $S_i, S_j \in \mathcal{S}$ *and* $i \neq j$ *imply* $S_i \cap S_j = \varnothing$
- *their union is* S*, that is,*

$$S = \cup_{S_i \in \mathcal{S}} S_i$$

2 Relations

2.1 Introduction

What does it mean to say that an object is "related" to another object? If the objects are people, then we understand what the term "related to" means (members of your family). When we are talking about abstract objects, then it is up to us as mathematicians to define what "related to" means.

Definition 13 (Binary Relation) *A binary relation* R *on two sets* A *and* B *is a subset of the Cross Product* $R \subseteq A \times B$

You should be familiar with many binary relations: $=, \leq, \geq, <, >, \subseteq$. For example the binary relation $\leq \subseteq \mathbb{N} \times \mathbb{N}$ is the set

$$\{(a, b) \mid a, b \in \mathbb{N} \text{ and } a \text{ is less than or equal to } b\}$$

2.2 Properties of Relations

Suppose R is a relation. We often write aRb to mean $(a, b) \in R$.

Suppose R is any relation on A and, that is $R \subseteq A \times A$. Suppose $a, b \in A$.

Reflexivity aRa for all $a \in A$

Symmetry if aRb then bRa

Antisymmetric if aRb and bRa then $a = b$

Transitive if aRb and bRc then aRc

Definition 14 (Equivalence Relation) *A relation R that is reflexive, symmetric and transitive is said to be an* **equivalence relation**

Definition 15 (Equivalence Class) *If R is an equivalence relation on A and B, then for each $a \in A$, the equivalence class of a, denoted by $[a]$ is the following set*

$$[a] = \{b \in B \mid aRb\}$$

Definition 16 (Partial Order) *A relation that is reflexive, antisymmetric and transitive is said to be a* **partial order***.*

The standard example of a partial order is the relation $\subseteq$.

The following is our first theorem. It is somewhat technical, but illustrates a fundamental idea about equivalence classes and partitions. Namely, that every partition has an equivalence relation associated with it, and every equivalence class has a partition associated with it.

Theorem 1 *The equivalence classes of any equivalence relation R on a set A forms a partition of A, and any partition of A determines an equivalence relation on A for which the sets in the partition are the equivalence classes.*

Proof Suppose R is an equivalence relation on A. We must show that the equivalence classes of R forms a partition of A.

1. Each equivalence class is non-empty, since aRa for all $a \in A$.

2. Clearly A is the union of all the equivalence classes (since each element of A belongs to at least one equivalence class)

3. We must show any two equivalence classes are disjoint. Let $[a], [b]$ be two distinct equivalence classes. Suppose $c \in [a] \cap [b]$. Then aRc and bRc. Hence by symmetry, cRb. And so by transitivity, aRb.

   Let $x \in [a]$, then xRc and by the above argument xRb (Why?), and so $x \in [b]$. Thus $[a] \subseteq [b]$. Using a similar argument, we can show $[b] \subseteq [a]$. Therefore $[a] = [b]$, which contradicts the fact that $[a]$ and $[b]$ are *distinct* equivalence classes.

For the second part of the theorem, suppose $\mathcal{A} = \{A_1, \ldots, A_n\}$ is any partition of A. Define $R = \{(a, b) \mid a \in A_i \text{ and } b \in A_i\}$. We must show that R is reflexive. Let $a \in A$ be any element. Then $a \in A_i$ for some i, and hence by definition of R, aRa. Next we will show that R is symmetric. Suppose aRb. Then $a \in A_i$ and $b \in A_i$ for some i. Then clearly, $b \in A_i$ and $a \in A_i$ and hence bRa. We must show R is transitive. Suppose, aRb and bRc. Then $a \in A_i$ and $b \in A_i$, and $b \in A_j$ and $c \in A_j$ for some i, j. Since $b \in A_i \cap A_j$, $A_i = A_j$ (since the elements of $\mathcal{A}$ are pairwise disjoint). Therefore, $a \in A_i$ and $c \in A_i$ and hence aRc.
QED

We will think of a function as a special type of relation:

Definition 17 (Function) *a function f is a binary relation on A and B such that for all $a \in A$, there exists a $b \in B$ such that $(a, b) \in f$. We will often write $f : A \to B$ and if $(a, b) \in f$, we will write $f(a) = b$.*

Suppose $f : A \to B$ is a function. A is said to be the **domain** and B the **codomain**.

Definition 18 (Image) *The image of a set $A' \subseteq A$ is the set:*

$$f(A') = \{b \mid b = f(a) \text{ for some } a \in A'\}$$

Definition 19 (Range) *The range of a function is the image of its domain.*

Suppose $f : A \to B$ is a function.

Definition 20 (Surjection) *f is a surjection (or onto) if its range is equal to its codomain. I.e., f is surjective iff for each $b \in B$, there exists an $a \in A$ such that $f(a) = b$*

Definition 21 (Injection) *f is an injection (or 1-1) if distinct elements of the domain produce distinct elements of the codomain. I.e., f is 1-1 iff $a \neq a'$ implies $f(a) \neq f(a')$, or equivalently $f(a) = f(a')$ implies $a = a'$.*

Definition 22 (Bijection) *f is a bijection if it is injective and surjective. In this case, f is often called a one-to-one correspondence.*

3 Proofs

3.1 Introduction

Learning how to write mathematical proofs takes time and hard work. One thing that must be stressed is knowing the formal definitions. A formal proof of a mathematical statement is simply an explanation of that statement *written in the language of mathematics*. If you don't know and understand the formal definitions, then you will not be able to write down your explanations. It would be like trying to explain something to someone in Italian without actually knowing the Italian language.

3.2 Proving Equality and Subset

How do you prove that two sets are equal? The answer to this question depends on who you are trying to convince. In this class, we will always error on the side of caution and given fairly detailed formal proofs. In turns out that proving two sets are equal reduces to proving the sets are subsets of each other.

Fact 1 *$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$*

Why is this true? Well, if A and B are equal, then they both name the same collection of objects. I.e., B is another name for the collection of objects that A names and vice versa. So, if A and B are equal then of course $A \subseteq B$ since A is always a subset of itself and B is simply another name for A. Similarly, we can show $B \subseteq A$. Conversely, suppose $A \subseteq B$ and $B \subseteq A$. We want to know that A and B name the same collection of objects. Suppose they didn't, then there should be some object $x \in A$ that is not in B **OR** some object $y \in B$ that is not in A. Well, we know the object x cannot exist since $A \subseteq B$ and so every element of A is an element of B. Similarly, the element y cannot exist. Hence, A and B must name the same collection of objects.

What about trying to prove that two sets are *not* equal? This turns out to be easier. In order to show that A does not equal B, you need only find an element in A that is not in B **OR** and element of B that is not in A. It will turn out that in general it is always easier to show a negative fact than a positive fact.

Showing two sets are equal reduces to proving that the sets are subsets of each other. But, how to show that a set is a subset of another set? The general procedure to show $A \subseteq B$ is to show that each element of A is also and element of

B. This is straightforward if A and B are both finite sets. For example, suppose $A = \{2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. How do we show that $A \subseteq B$. Since A is finite, we simply notice that $2 \in B$, $3 \in B$ and $4 \in B$.

What if A is the set of even numbers and B is the set of all integers? We would get awfully tired (and bored) if we waited around to show that each and every element of A is also an element of B. Imagine A and B are two boxes, and you would like to know whether all the elements in A's box are also in B's box. Suppose you reach in box A and select an element say 10. After inspecting 10, you notice that 10 is in fact an integer and so must also be an element of box B. But you are not satisfied, since you cannot be sure that the next element you choose from A will also be an element of B. In fact, even if you have shown that the first million even integers are all members of box B, you cannot be sure that the next element you select from box A will in fact be an integer. Instead you should consider the *property* that x satisfies when it is a member of A, and show that it must be the case that x is an element of B. What property does x satisfy if it is contained in A's box? The answer is $x = 2 \cdot n$, where n is some integer. Then you simply notice that if n is an integer, then $2 \cdot n$ is also an integer; and hence, x is an element of B.

3.3 Examples

Theorem 2 $\overline{A} \cup \overline{B} = \overline{A \cap B}$

Proof We must show $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$ and $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

We will show $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. Suppose $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ **OR** $x \in \overline{B}$. Suppose $x \in \overline{A}$ then $x \notin A$. Then $x \notin A \cap B$ (if x is not in A then x is certainly not in both A and B). Hence $x \in \overline{A \cap B}$. Suppose $x \in \overline{B}$. For similar reason, $x \in \overline{A \cap B}$. Hence in either case, $x \in \overline{A \cap B}$. Therefore, $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

We must show $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Suppose $x \in \overline{A \cap B}$. Then $x \notin A \cap B$, and so $x \notin A$ **OR**[2] $x \notin B$. Hence either $x \in \overline{A}$ or $x \in \overline{B}$. In either case, $x \in \overline{A} \cup \overline{B}$.
QED

[2]NOTICE that $x \notin A \cap B$ **DOES NOT IMPLY** $x \notin A$ *and* $x \notin B$. The "and" in italics is shold be an "or". Make sure you clearly understand the logic here, since this is often misunderstood by students.

Theorem 3 *$A \subseteq B$ iff $A \cap B = A$.*

Proof We must show $A \subseteq B$ implies $A \cap B = A$ **AND** $A \cap B = A$ implies $A \subseteq B$.

Assume that $A \subseteq B$. We must show $A \cap B = A$. I.e. we must show (1) $A \cap B \subseteq A$ and (2) $A \subseteq A \cap B$. The first statement is trivial, it is always the case that $A \cap B \subseteq A$. For the second statement, assume $x \in A$. We must show $x \in A \cap B$. Since $A \subseteq B$, $x \in B$. Hence $x \in A \cap B$.

Assume $A \cap B = A$. We must show $A \subseteq B$. Let $x \in A$. Then $x \in A \cap B$ since $A = A \cap B$. Hence $x \in B$.
QED